ON A MOTION OF AN EQUILIBRATED GYROSCOPE IN THE NEWTONIAN FORCE FIELD

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1. It is well known [1,2] that the equations of motion of a heavy solid about a point fixed in its center of gravity in the Newtonian force field are

$$A\frac{dp}{dt} + (C-B)\,qr = 3\frac{g}{R}(C-B)\,\gamma'\gamma'', \quad \frac{d\gamma}{dt} = r\gamma' - q\gamma'' \begin{pmatrix} ABC \\ pqr \\ \gamma\gamma'\gamma'' \end{pmatrix} \qquad (1.1)$$

with the particular solution

$$\gamma = A'\alpha^{-1/2}p, \qquad \gamma' = B'\alpha^{-1/2}q, \qquad \gamma'' = C'\alpha^{-1/2}r \qquad (\alpha = 3g/R) \qquad (1.2)$$

where A', B' and C' are constants to be determined. Introducing in (1.1) the dimensionless variables

$$p_1 = \alpha^{-1/2}p, \qquad q_1 = \alpha^{-1/2}q, \qquad r_1 = \alpha^{-1/2}r, \qquad t_1 = \alpha^{1/2}t \qquad (1.3)$$

and taking into account (1.2) we obtain the equations

$$A\frac{dp_1}{dt_1} + (C-B)(1-B'C')\,q_1r_1 = 0, \qquad A'\frac{dp_1}{dt_1} + (C'-B')\,q_1r_1 = 0 \qquad \begin{pmatrix} ABC \\ A'B'C' \\ p_1q_1r_1 \end{pmatrix} \qquad (1.4)$$

from which we are able to determine A', B' and C' as

$$\frac{C-B}{A}u + v - w = \frac{C-B}{A}, \qquad A' = \sqrt{\frac{u}{vw}}\,\mathrm{sgn}\,u \qquad \begin{pmatrix} ABC \\ A'B'C' \\ uvw \end{pmatrix} \qquad (1.5)$$

Since the determinant of the linear system (1.5) equals zero and the system is consistent, its solution

$$u = 1 - a + wa, \quad v = 1 - b + wb \quad (a = A/C,\ b = B/C,\ a + b \neq 1) \quad (1.6)$$

depends on one arbitrary constant w.

Let us consider the following case:

$$u > 0, \quad v > 0, \quad w > 0; \qquad A' > B' > C' \quad (1.7)$$

which, by (1.5) occurs under the condition $u > v > w$ which is satisfied, in particular, when

$$w < 1, \qquad C > B > A \quad (1.8)$$

Then the second group of equations in (1.4) can be taken as the classical equations of motion of a solid about a fixed point in the case of Euler where A', B' and C' are moments of inertia. There equations possess the first integrals

$$A'p_1^2 + B'q_1^2 + C'r_1^2 = D^{-1}, \qquad A'^2p_1^2 + B'^2q_1^2 + C'^2r_1^2 = 1 \quad (1.9)$$

where D is an arbitrary constant, and the second constant of integration is set to be equal to unity (in the conventional Greenhill notation $\mu^2D^2 = 1$) because of the trivial integral of system (1.1)

$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1$$

Assuming that $B' > D$, we can express [3] the formulas for p_1, q_1 and r_1 in the form

$$p_1 = -\sqrt{\frac{D - C'}{DA'(A' - C')}}\operatorname{cn}\tau, \quad q_1 = \sqrt{\frac{D - C'}{DB'(B' - C')}}\operatorname{sn}\tau, \quad r_1 = \sqrt{\frac{A' - D}{DC'(A' - C')}}\operatorname{dn}\tau$$

$$\tau = n(t_1 - t_0), \qquad n^2 = \frac{(A' - D)(B' - C')}{DA'B'C'}, \qquad k^2 = \frac{(A' - B')(D - C')}{(B' - C')(A' - D)} \quad (1.10)$$

These formulas, together with (1.2) and (1.3), give the solution of the problem.

2. Let us assume that r_{10} is large and that at the initial instant of time the following inequality

$$0 < \gamma_0'' < 1 \qquad (f(t_0) = f_0) \quad (2.1)$$

is satisfied.

Further, assuming that $\gamma_0' = 0$, we obtain from formulas (1.10) that $t_0 = 0$.

Let us introduce the small parameter

$$\lambda = \gamma_0'' / r_{10} \qquad (2.2)$$

The quantities A', B', C', D, u, v, w, n and k will be expressed as power series in λ

$$u = 1 - a + \lambda^2 (\ldots), \quad v = 1 - b + \lambda^2 (\ldots), w = \lambda^2 (1 - a)(1 - b) + \lambda^4 (\ldots) \qquad (2.3)$$
$$A' = [\lambda (1 - b)]^{-1} + \lambda (\ldots), \quad B' = [\lambda (1 - a)]^{-1} + \lambda (\ldots), \quad c' = \lambda, \quad D = \lambda \gamma_0''^{-2} + \lambda^3 (\ldots)$$
$$n = \frac{\gamma_0''}{\lambda} + \lambda (\ldots), \qquad k^2 = \lambda^2 \frac{(1 - \gamma_0''^2)(b - a)}{\gamma_0''^2} + \lambda^4 (\ldots)$$

While investigating the motion of a rigid body we shall use the Eulerian angles

$$\cos\theta = \gamma'', \qquad \frac{d\psi}{dt} = \frac{p\gamma + q\gamma'}{1 - \gamma''^2}, \qquad \frac{d\varphi}{dt} = r - \frac{d\psi}{dt}\cos\theta \qquad \left(\tan\varphi_0 = \frac{\gamma_0}{\gamma_0'}\right) \qquad (2.4)$$

By (1.2), (1.3), (1.9) and (1.10) the first formula in (2.4) can be reduced to $\cos\theta = \gamma_0''\,\mathrm{dn}\,\tau$, and by retaining only the first two terms in the expansion of $\mathrm{dn}\,\tau$ in series of powers of the small parameter k^2, we obtain by (2.3)

$$\cos\theta = \gamma_0'' \left[1 - \lambda^2 \frac{(1 - \gamma_0''^2)(b - a)}{4\gamma_0''^2} (1 - \cos 2r_0 t) + \ldots\right] \qquad (2.5)$$

By (1.2), (1.3), (1.9), (1.10) and (2.3) we can reduce the second equation in (2.4) to

$$\frac{d\psi}{d\tau} = \delta\left(1 + \frac{\beta}{1 + \nu\,\mathrm{sn}^2\tau}\right), \qquad \beta = \frac{\lambda D^{-1} - 1}{1 - \gamma_0''^2} = -1 + \lambda^2 (1 - b) + \ldots$$
$$\nu = k^2 m, \qquad m = \frac{\gamma_0''^2}{1 - \gamma_0''^2}, \qquad \delta = \frac{1}{\lambda n} = \gamma_0''^{-1} + \lambda (\ldots) \qquad (2.6)$$

Integrating (2.6) we obtain [4]

$$\psi - \psi_0 = \delta\left[J_1 \tau + J_2 \ln \frac{\theta(\tau - \xi)}{\theta(\tau + \xi)}\right], \qquad \mathrm{sn}^2 \xi = -m$$
$$J_1 = 1 + \beta\left[1 + \frac{\mathrm{sn}\,\xi}{\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi} \frac{\theta'(\xi)}{\theta(\xi)}\right], \qquad J_2 = \frac{\beta\,\mathrm{sn}\,\xi}{2\,\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi} \qquad (2.7)$$

Let us set

$$\theta(\tau + \xi) = \rho e^{i\chi}, \quad \theta(\tau - \xi) = \rho e^{-i\chi} \qquad \left(\ln \frac{\theta(\tau - \xi)}{\theta(\tau + \xi)} = -2i\chi\right) \qquad (2.8)$$

Retaining only the first terms in the expansion in powers of λ in (2.7) we have

$$\frac{\mathrm{sn}\,\xi}{\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi} \frac{\theta'(\xi)}{\theta(\xi)} = -\frac{1}{2} k^2 m + \ldots, \chi = \frac{k^2}{4} \sqrt{m(m + 1)} \sin 2\tau + \ldots$$
$$J_1 = \frac{1}{2} \lambda^2 (2 - a - b) + \ldots, \qquad J_2 = \frac{i}{2} \left(\frac{m}{m + 1}\right)^{1/2} \qquad (2.9)$$

By (2.3), (2.8) and (2.9) the formula for the precession angle (2.7) takes the form

$$\psi - \psi_0 = \frac{1}{2}\alpha^{1/2}\lambda\,(2 - a - b)\,t - \lambda^2\,\frac{b-a}{4\gamma_0''^2}\sin 2r_0 t + \ldots \qquad (2.10)$$

By (1.10), (2.5) and (2.10) we obtain from the last two formulas in (2.4) the following expression for the angle of natural rotation:

$$\varphi - \frac{\pi}{2} = \left\{r_0 - \frac{\alpha}{4r_0}\,[b - a + \gamma_0''^2(4 - a - 3b)]\right\} t + \ldots \qquad (2.11)$$

In formulas (2.5), (2.10) and (2.11) three arbitrary constants appear: ψ_0, $\cos\theta_0 = \gamma_0''$ and r_0 (r_0 is large). By substituting $t + h$ for t (system (1.1) is autonomous) we can add a fourth arbitrary constant φ_0, which is related to h on the strength of (2.11), through the following formula:

$$\varphi_0 = \frac{\pi}{2} + r_0 h + \ldots \qquad (2.12)$$

Let us note that formulas (2.5), (2.10) and (2.11) which determine the angles θ, ψ and φ differ substantially from the corresponding formulas in the case of Euler. Indeed, if in the case of Euler we introduce the small parameter $\lambda = \gamma_0''/r_0$, then D will not depend on λ. Consequently, the quantity k^2 will also be independent of λ and the Jacobi functions sn τ, cn τ and dn τ cannot be expanded in powers of λ.

3. Let us now consider the motion of a rigid body, using (2.5), (2.10) and (2.11); formulas (2.5) and (2.10) will be put in the form

$$\theta - \theta_{10} = -\,s\sin\theta_0\cos 2r_0\,(t + h) + \ldots, \qquad \theta_{10} = \theta_0 + s\sin\theta_0 +$$

$$\psi - \psi_0 = \frac{1}{2}\alpha^{1/2}\lambda\,(2 - a - b)\,t - s\sin 2r_0\,(t + h) + \ldots, \quad (s = \lambda^2\,(b - a)/4\cos\theta_0) \qquad (3.1)$$

Consider a unit sphere, centered on the fixed point, and form on its surface a spherical rectangle made up of two parallels distant from the middle parallel θ_{10} by the angles $\pm\, s\sin\theta_0$ and of two meridians distant from the middle meridian ψ_0 by the angles $\pm\, s$. Then, the trajectory (θ_1, ψ_1) traced by the z-axis on our unit sphere rotating with the constant angular velocity $\dot\psi_1 = 1/2\, s^{1/2} 1\,(2 - a - b)$ about the fixed axis z_1, is the ellipse

$$\frac{(\theta - \theta_{10})^2}{(s\sin\theta_0)^2} + \frac{(\psi_1 - \psi_0)^2}{s^2} = 1 \qquad (3.2)$$

which is tangent to the spherical rectangle at the midpoint of its sides. When tracing this ellipse, the z-axis performs in the first approximation a periodic motion with the period $T = \pi/r_0$, and at the instants of time

$$t_1^* = \frac{\pi\,(n_1 + {}^3/_2) - 2\varphi_0}{2r_0}, \qquad t_2^* = \frac{\pi\,(n_1 + 1) - 2\varphi_0}{2r_0} \qquad (n_1 = 0, \pm 1, \pm 2, \ldots) \qquad (3.3)$$

the z-axis will pass through the points of intersection of the middle parallel with the outer meridians (t_1^*) and through the points of intersection of the middle meridian with the outer parallels (t_2^*).

The natural rotation of the body as shown by formula (2.11) differs very little from the uniform rotation with large angular velocity r_0.

Let us consider now the cases $\gamma_0'' = 1$ and $\gamma_0'' = 0$. When $\gamma_0'' = 1$ ($\gamma_0 = \gamma_0' = 0$) formulas (1.2) show that $p_0 = q_0 = 0$ ($A' \neq 0$, $B' \neq 0$). Then, from the integrals (1.9) follows that $p = q = 0$ ($\gamma = 0$, $\gamma' = 0$) at all values of time, and the body rotates with the constant angular velocity r_0 about the fixed axis z_1.

The case $\gamma_0'' = 0$ reduces to the case $r_0 = 0$ (if $C' \neq 0$) which is of no interest, or (if $C' = 0$) to the case $p = q = \gamma'' = 0$, which has been investigated under more general conditions in [1].

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